Student Name: ______________________

Teacher: ___________________________

Sydney Technical High School

2023 | HIGHER SCHOOL CERTIFICATE **TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen.
- Calculators approved by NESA may be used.
- A reference sheet is provided.
- For questions in Section II, show relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless work or illegible writing.
- **Begin each question on a new page.**

Total Marks: 100

Section I – 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II – 90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 The magnitude of the vector $\underset{\sim}{u} = a\underset{\sim}{i} - 6\underset{\sim}{j} + 8\underset{\sim}{k}$ is 12.

What is a possible value of *a*?

A. $2\sqrt{11}$

B. $11\sqrt{2}$

C. 44

D. 2

2 Which expression below is equivalent to: $\dfrac{4e^{-\left(\frac{5i\pi}{6}\right)}}{2e^{\left(\frac{i\pi}{2}\right)}}$?

A. $-1-\sqrt{3}i$

B. $-1+\sqrt{3}i$

C. $1-\sqrt{3}i$

D. $1+\sqrt{3}i$

3 Choose one of the statements to describe the relationship between P and Q.
P : $n = 8$ Q : $n^2 = 64$

A. $P \Leftrightarrow Q$

B. $P \Leftarrow Q$

C. $P \Rightarrow Q$

D. $\neg P \Leftarrow Q$

4 The vectors $\underset{\sim}{u} = 6\underset{\sim}{i} - 2\underset{\sim}{j} + p\underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} + p\underset{\sim}{j} + 3\underset{\sim}{k}$ are perpendicular.
What is the value of *p*?

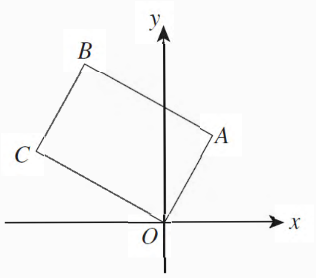
A. -12

B. $-\frac{12}{5}$

C. $\frac{12}{5}$

D. 12

5 The Argand diagram shows the rectangle *OABC* where OC = 4OA. Vertex *A* corresponds to the complex number *w*.
Which of the following complex numbers corresponds to vertex C?



A. $-4iw$

B. $4iw$

C. $-4w$

D. $4w$

6 A particle is moving along the $x - axis$ in SHM. The displacement of the is x –metres, The particle is at rest when $x = -2$ and again, when $x = 8$. It takes 6 seconds for the particle to move from x = - 2 to $x = 8$.
What is the maximum velocity of the particle?

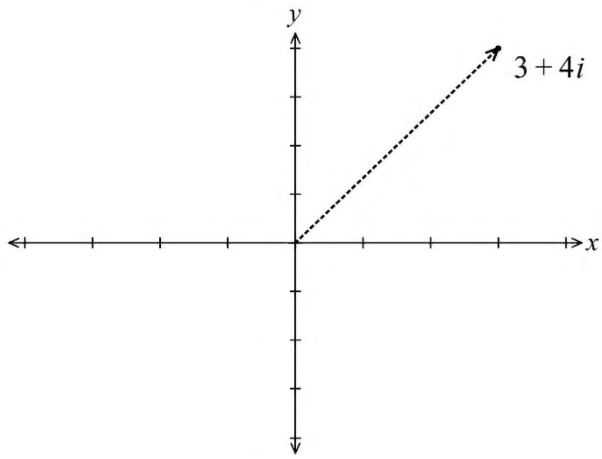
A. $\frac{5\pi}{6}$ m/s

B. $\frac{4\pi}{3}$ m/s

C. $\frac{5\pi}{3}$ m/s

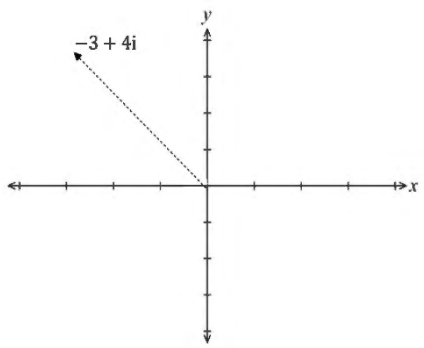
D. $\frac{10\pi}{3}$ m/s

7 The complex number $z = 3 + 4i$ is shown.

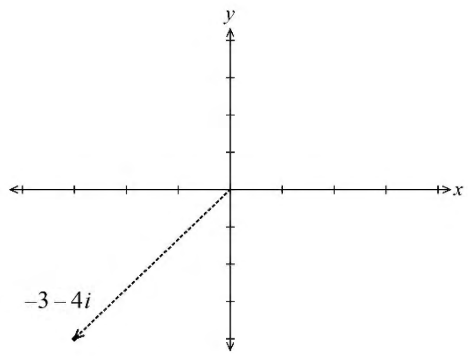


Which of the following is the graph of $i^2\bar{z}$?

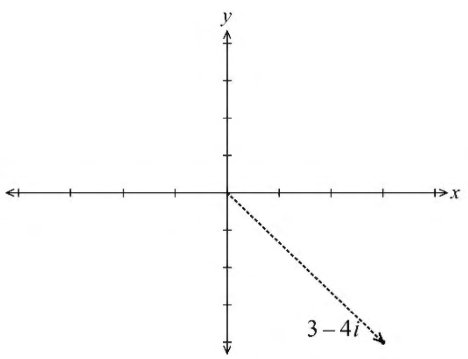
A.



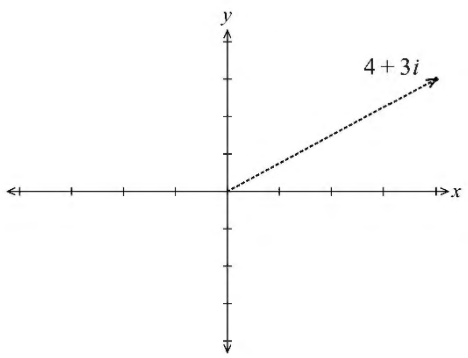
B.



C.



D.



8 The two possible values of $Re(\sqrt{i}) + |Im(\sqrt{i})|$ are:

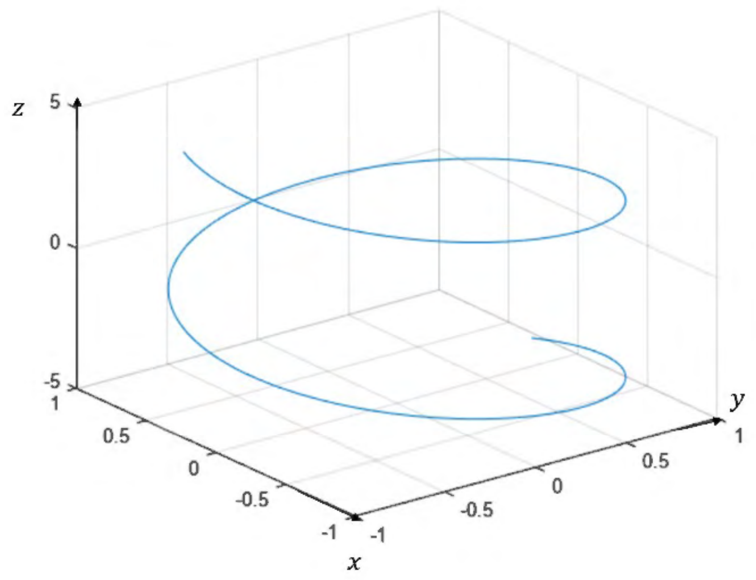
A. $-1, 1$

B. $0, \sqrt{2}$

C. $0, \ 1$

D. $1, \sqrt{2}$

9 The parametric equation of the curve shown is:



A. $x = t^2\ , y = t,\ z = 1$

B. $x = sint\ , y = cos2t,\ z = t$

C. $x = sint\ , y = cost,\ z = 1$

D. $x = sint\ , y = cost,\ z = t$

10 The value of

$$\int_1^5 (x-1)\sqrt{5-x}\,dx$$

A. $\frac{96}{5}$

B. $\frac{16}{3}$

C. $\frac{208}{15}$

D. $\frac{128}{15}$

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section.

Answer each question in the writing booklet. Extra writing paper is available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start this question at the top of a **NEW** page.

(a) Express $\frac{2+i}{3-i}$ in the form $x + iy$, where x and y are real numbers. **2**

(b) Find the value of m given that i is a root of the equation,

$z^2 + mz + (1 - i) = 0$ **2**

(c) i. Write the complex number $-\sqrt{3} - i$ in exponential form. **2**

ii. Hence, find the exact value of $\left(-\sqrt{3} - i\right)^{16}$ in the form $x + iy$. **2**

(d) By choosing a suitable substitution, or otherwise, evaluate, **4**

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1 - sin2x}\,(1 - 2\,cos^2 x)\,dx$$

(e) Find, $\int \sin^2 2x\, sinx\;\, dx$. **3**

Question 12 (15 marks) Start this question at the top of a **NEW** page.

(a) For integer n, use the contrapositive, or otherwise, to prove the following statement:

"If n^2 is even, then n is even". **3**

(b) A triangle is formed in three – dimensional space with vertices *A*(1, -2, 3) *B*(2, 3, 1) and *C*(-1, 3, 2)

Find the size of $\angle ABC,$ correct to the nearest degree. **3**

(c) Use the substitution $t = tanx$, and find: **3**

$$\int \frac{dx}{3\cos^2 x + \sin^2 x}$$

(d) Let l_1 be the line with equation $\underset{\sim}{r} = (-\underset{\sim}{i} + 2\underset{\sim}{j}) + \lambda(2\underset{\sim}{i} + 5\underset{\sim}{j}),\ \lambda \in \mathbb{R}$.

The line l_2 passes through the point A(1, -2) and is parallel to l_1.

Find the equations of l_2 in the form $y = mx + c$. **3**

(e) Let $a = \omega$ and $b = \omega^2$ be the two complex cube roots of unity. **3**

If $x\ =\ 9a + 5b$ and $y\ =\ 5a + 9b$, then evaluate xy.

Question 13 (15 marks) Start this question at the top of a **NEW** page.

(a) i. Decompose $\frac{1}{(x^2)(2x-1)}$ into partial fractions. **3**

ii. Hence, $\int \frac{1}{(x^2)(2x-1)}\, dx$. **2**

(b) Find the least positive integer k such that $\cos\left(\frac{4\pi}{7}\right) + isin\left(\frac{4\pi}{7}\right)$ is a solution to $z^k = 1$. **2**

(c) Show that the complex number w is a solution of $z^n = 1$, then so is w^m, where m and n are both arbitrary integers. **2**

(d) The probability density function describing the likelihood of a continuous variable occurring is given by the function.

$$f(x) = \begin{cases} k(sinx + cosx) & -\frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \\ 0 & elsewhere \end{cases}$$

(i) For the domain of $-\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$, express $f(x)$ in the form $Rsin(x + \alpha)$, where $R > 0$ and α is acute. **2**

(ii) Calculate the mean of this distribution. **2**

(iii) Calculate the variance of this distribution. **2**

Question 14 (15 marks) Start this question at the top of a **NEW** page.

(a) Write the negation of the statement P: “I am beautiful and tall” **2**

(b) Consider the complex number $z = cos\theta + isin\theta$

(i) Using De Moivre’s theorem, show that.

$z^n + \frac{1}{z^n} = 2cosn\theta,$ for $n \in \boldsymbol{Z}$ **2**

(ii) Hence or otherwise express $(z + \frac{1}{z})^6$ in the form

$Acos6\theta + Bcos4\theta + Ccos2\theta + D$ where $A, B, C, D \in \boldsymbol{R}$ **2**

(iii) Hence, evaluate $\int_0^{\frac{\pi}{6}} \cos^6 \theta \, d\theta$ **2**

(c) A particle is oscillating in simple harmonic motion such that its displacement x metres from a given O satisfies the equation $\ddot{x} = -4(x + 2)$, where t is the time in seconds.

(i) If the particle is initially at rest at $x = 2$, show that the speed *v* m/s of a particle moving along the *x*-axis is given by $v^2 = 48 - 16x - 4x^2$. **2**

(ii) Find the displacement of the particle after t seconds of motion. **3**

(d) Sketch the graph of $\left|z - \sqrt{2}e^{\frac{\pi}{4}i}\right| < 2$ on an argand diagram. **2**

Question 15 (15 marks) Start this question at the top of a **NEW** page.

(a)

(i) By choosing a suitable substitution, show that, **2**

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

(ii) Hence, or otherwise, determine the value of. **2**

$\int_0^1 x^2\sqrt{1-x}dx$

(b) A particle of mass 4 kg moves in a straight line such that at time t seconds its

Displacement from a fixed origin is x metres and its speed in $v\,ms^{-1}$.

The resultant force F is given as $F = 16 - 4v^2\ N$.

(i) Find an expression for x as a function of v, given that the particle starts at the origin with $v = 0\ ms^{-1}$. **2**

(ii) Find an expression for v as a function of t and hence find the displacement of the particle after 3 seconds. **4**

(c) The sequence $\{x_n\}$ is given by $x_1 = 1$ and $x_{n+1} = \frac{4+x_n}{1+x_n}$ for $n \geq 1$

(i) Prove by Mathematical Induction that for $n \geq 1$, **4**

$x_n = 2\left(\frac{1+\alpha^n}{1-\alpha^n}\right)$ where $\alpha = -\frac{1}{3}$

(ii) Hence find the limiting value of x_n as $n \to \infty$ **1**

Question 16 (15 marks) Start this question at the top of a **NEW** page.

(a) Given that n is an integer that is **not** divisible by 3, prove that $n^2 + 6n + 2$ is a multiple of 3. **4**

(b) Let $I_n = \int_0^1 (1 - x^2)^n \, dx$ and $J_n = \int_0^1 x^2(1 - x^2)^n \, dx$.

(i) Show that $I_n = 2nJ_{n-1}$ **2**

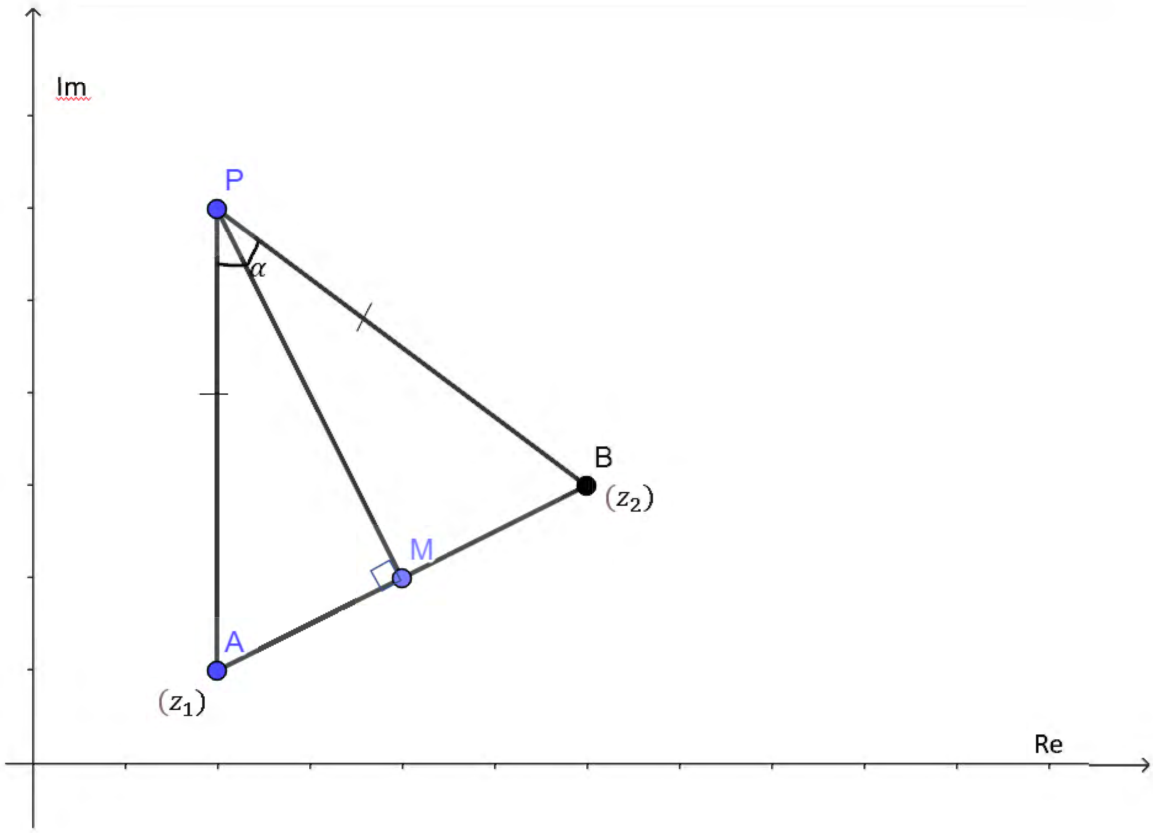
(ii) Show that $I_n = \frac{2n}{2n+1} I_{n-1}$ **2**

(iii) Show that $J_n = I_n - I_{n+1}$, and hence deduce that $J_n = \frac{1}{2n+3} I_n$ **2**

(iv) Hence write down a reduction formula for J_n in terms of J_{n-1}. **1**

Question 16 continues on the next page…….

(c) The diagram shows isosceles triangle *ABP* where *AP* = *BP* and $\angle APB = \alpha$.
PM is the altitude of triangle.



Suppose that A and B represent the complex numbers z_1 and z_2 respectively.

(i) Find the complex number represented by: **2**

α. $\overrightarrow{AM}$

β. $\overrightarrow{MP}$

(ii) Hence show that P represents the complex number: **2**

$$\frac{1}{2}\left(1 - i\cot\frac{\alpha}{2}\right)z_1 + \frac{1}{2}\left(1 + i\cot\frac{\alpha}{2}\right)z_2$$

END OF TASK

Section I

1. A 2. B 3. C 4. A 5. B
6. A 7. A 8. B 9. D 10. D.

Solutions: multiple choice

1. $\sqrt{a^2+36+64} = 12$
$a^2+100 = 144$
$a = 2\sqrt{11}$ (A)

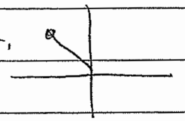
2. $2e^{\left(-\frac{5\pi}{2}-\frac{\pi}{2}\right)i} = 2e^{2\pi/3\, i}$
$= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
$= 2\left(-\frac{1}{2} + i\sqrt{3}/2\right)$
$= -1+\sqrt{3}i$ (B)

3. $n=8 \quad \therefore n^2 = 8^2$
but $n^2 = 64 \quad n = \pm 8$
$\therefore P \Rightarrow Q$ (C)

4. $\underset{\sim}{u}\cdot\underset{\sim}{v} = 0 \quad 12-2p+3p = 0$
$p = -12$ (A)

5.  $90°$ and $\times 4$ $\therefore \vec{OC} = 4|w|$ after rotation
$= 4i\omega$. (B)

6. $-2 \quad 3 \quad 8$ max vel @ centre
$a = 5$ $\quad 2\pi/n = 12 \quad n = \pi/6$
using $v^2 = n^2\left(a^2-(x-a)^2\right)$
$= \pi^2/36\left(25-(x-3)^2\right)$ at $x=3$
$v^2 = 25\pi^2/36 \quad \therefore v = 5\pi/6$ (A)

7.  $i^2 = -1$
$\bar{z} = 3-4i \quad \therefore -3+4i$ (A)

8. $\sqrt{i} = x+iy \rightarrow i = x^2+2xyi-y^2$
$\therefore x^2-y^2 = 0 \quad 2xy = 1 \quad \therefore x = \pm 1/\sqrt{2}$
$x = y \quad xy = 1/2 \quad y = \pm 1/\sqrt{2}$
$\therefore 1/\sqrt{2}+1/\sqrt{2} = \sqrt{2}$ or $-1/\sqrt{2}+\left|-1/\sqrt{2}\right| = 0$

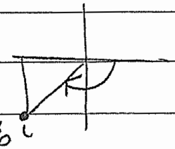
9. (D)

10. $\int^5 (x-1)\sqrt{5-x}\,dx$
let $u = 5-x$ $\quad \int_4^0 (5-u-1)\sqrt{u}\,\cdot -du$
$x = 5-u$
$-\int_4^0 4\sqrt{u}-u\sqrt{u}\,du$
$\rightarrow \int_0^4 4u^{1/2}-u^{3/2}\,du$
$\left[\frac{2}{3}4u^{3/2}-\frac{2}{5}u^{5/2}\right]_0^4$
$= \frac{8}{3}\cdot 8 - \frac{2}{5}\cdot 32 - 0$
$= 128/15$ (D)

Question 11

a) $\frac{2+i}{3-i} \times \frac{3+i}{3+i}$
$= \frac{6+5i-1}{9+1}$
$= \frac{5+5i}{10}$
$= 1/2 + 1/2\, i$

b) $i^2 + mi + (1-i) = 0$
$-1 + i(m-1) + 1 = 0$
$m = 1$

c) $-\sqrt{3}-i$

$= 2e^{-5\pi/6\, i}$

ii) $\left(2e^{-5\pi/6\, i}\right)^{16}$
$= 2^{16}e^{-40\pi/3\, i}$
$= 2^{16}e^{2\pi/3\, i}$
$= 2^{16}\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$
$= 2^{16}\left[-1/2 + i\sqrt{3}/2\right]$
$= 2^{15}\left(-1+\sqrt{3}i\right)$

d) $\int_{\pi/4}^{\pi/2} \sqrt{1-\sin 2x}\,\left(1-2\cos^2 x\right)dx$
i)
let $u = 1-\sin 2x$
$du = -2\cos 2x\,dx$
$\frac{du}{2} = \left(1-2\cos^2 x\right)dx$

$\int_0^1 \sqrt{u}\cdot\frac{du}{2}$

$1/2\int_0^1 u^{1/2}\,du$

$x = \pi/2 \quad u = 1-\sin 0$
$= 1$

$= 1/2 \cdot 2/3\, u^{3/2}\Big]_0^1$

$x = \pi/4 \quad u = 1-\sin\pi/2$
$= 0$

$= \frac{1}{3}(1-0)$
$= 1/3$.

Q11

d) (ii) $\int \sin^2 2x \sin x \, dx$ $\qquad \sin 2x = 2\sin x\cos x$

$= \int 4\sin^2 x \cos^2 x \sin x \, dx$

$= 4\int \cos^2 x \sin^3 x \, dx$

$= 4\int \cos^2 x (1-\cos^2 x)\sin x \, dx$

$= 4\int u^2(1-u^2) \cdot -du$ $\qquad$ let $u = \cos x$, $du = -\sin x \, dx$

$= 4\int u^2(u^2-1)\, du$

$= 4\left[\frac{u^5}{5} - \frac{u^3}{3}\right]$

$= \frac{4}{5}\cos^5 x - \frac{4}{3}\cos^3 x + C$

Question 12

a) Contrapositive statement: if n is odd then n^2 is also odd

let n be an odd number where $n = 2p+1$, $p \in \mathbb{Z}^+$

Now $n^2 = (2p+1)^2$

$= 4p^2 + 4p + 1$

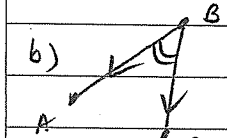
$= 2(p^2+2p)+1$

$= 2q+1$ where $q = 2p^2+2p$

which is odd

$\therefore$ If n is odd then n^2 is also odd

ie if n^2 is even then n is even.

b) 

$\overrightarrow{BA} = \begin{pmatrix} -1 \\ -5 \\ 2 \end{pmatrix}$ $\qquad \overrightarrow{BC} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = \sqrt{1+25+4} \times \sqrt{9+0+1} \cos\theta$

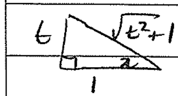
$3+0+2 = \sqrt{30}\cdot\sqrt{10}\cos\theta$

$5 = 10\sqrt{3}\cos\theta$

$\cos\theta = 1/2\sqrt{3}$

$\theta \doteqdot 73 \cdot 221\ldots \simeq 73°$

c) $t = \tan x$



$dt = \sec^2 x \, dx$

$\cos^2 x \, dt = dx$

$\frac{1}{t^2+1} dt = dx$

Now $\int \frac{1}{\frac{3}{t^2+1} + \frac{t^2}{t^2+1}} \cdot \frac{1}{t^2+1} dt$

$= \int \frac{1}{3+t^2} dt$

$= 1/\sqrt{3} \tan^{-1}(t/\sqrt{3})$

$= 1/\sqrt{3} \tan^{-1}\left[\frac{\tan x}{\sqrt{3}}\right] + C$

d) parallel ∴ same direction vector

$$\ell_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$x = 1 + 2\mu \quad \therefore \mu = \dfrac{x-1}{2}$

$y = -2 + 5\mu$

and $y = -2 + \left(\dfrac{x-1}{2}\right) \cdot 5$

$y = -2 + \dfrac{5x}{2} - \dfrac{5}{2}$

$y = \dfrac{5x}{2} - \dfrac{9}{4}$

e) ω and ω^2 are the 2 complex cube roots of unity

$\boxed{\omega^3 = 1}$ $\qquad \therefore a = \omega$

$(\omega - 1)(\omega^2 + \omega + 1) = 0 \qquad b = \omega^2$

Real sol^n $\qquad \therefore \boxed{\omega^2 + \omega = -1} \qquad b^2 = \omega^4$

$= \omega \times \omega^3$

$xy = (9a + 5b)(5a + 9b)$

$= 45a^2 + 81ab + 25ab + 45b^2$

$= 45(a^2 + b^2) + 106ab \qquad$ Now $a = \omega \quad b = \omega^2$

$= 45(\omega^2 + \omega^4) + 106\,\omega \cdot \omega^2 \qquad \omega^3 = 1$

$= 45(\omega^2 + \omega) + 106$

$= 45(-1) + 106$

$= 61$

Question 13

a) $\dfrac{1}{x^2(2x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{2x-1}$

$\therefore 1 = A(x)(2x-1) + B(2x-1) + C(x^2)$

let $x = 0$

$1 = B(-1) \qquad B = -1$

let $x = 1/2$

$1 = C(1/2)^2 \qquad \therefore C = 4$

let $x = -1$ OR coeff of x^2

$0 = 2A + C$

$0 = 2A + 4 \qquad \therefore A = -2$

ii) $\int f(x)\,dx = \int \dfrac{-2}{x} - \dfrac{1}{x^2} + \dfrac{4}{2x-1}\,dx$

$= -2\ln|x| + \dfrac{1}{x} + 2\ln|2x-1| + C$

$= \dfrac{1}{x} + 2\ln\left|\dfrac{2x-1}{x}\right| + C$

b) i) $z^k = cis(0 + 2\pi n) \qquad n \in \mathbb{Z}$

$\therefore z = cis\left(\dfrac{0 + 2\pi n}{k}\right)$

when $n = 2$, $k = 7 \qquad cis\left(\dfrac{4\pi}{7}\right)$ is a sol^n

$\therefore k = 7$ is the smallest integer value

c) if ω is a sol^n to $z^n = 1$

then $\omega^n = 1$

Now $(\omega^n)^m = 1^m \qquad 1^m = 1$

$\omega^{nm} = 1$

$(\omega^m)^n = 1$

$\therefore \omega^m$ is a sol^n to $z^n = 1$.

d) $k(\sin x + \cos x) \doteq R\sin(x+\alpha)$

$\alpha = \pi/4$ by inspection (aux ∠ technique)

$\int_{-\pi/4}^{3\pi/4} R\sin(x+\alpha) = 1$ as a PDF

$\left.-\cos\left(x+\frac{\pi}{4}\right)\right]_{-\pi/4}^{\frac{3\pi}{4}} = 1/R$

$-\cos\left(\frac{3\pi}{4}+\frac{\pi}{4}\right) - \left[-\cos 0\right] = 1/R$

$-(-1) + 1 = 1/R$

$2 = 1/R$

$R = 1/2$

$\therefore f(x) = 1/2\sin(x+\pi/4) \quad -\pi/4 \le x \le 3\pi/4.$

ii) $\mu = \int x f(x)\,dx$

$= \int_{-\pi/4}^{3\pi/4} 1/2\, x\sin(x+\pi/4)\,dx$

$= 1/2\left[-x\cos(x+\pi/4) - \int -1\cos(x+\pi/4)\,dx\right]$

$= \frac{1}{2}\left[-x\cos(x+\pi/4) + \int\cos(x+\pi/4)\,dx\right]$

$= 1/2\left[-x\cos(x+\pi/4) + \sin\left(x+\frac{\pi}{4}\right)\right]_{-\pi/4}^{3\pi/4}$

$= \frac{1}{2}\left[-\left(\frac{3\pi}{4}\right)\cos\pi + 0 - \left[-\left(\frac{-\pi}{4}\right)\cos 0 + 0\right]\right]$

$= \frac{1}{2}\left[3\pi/4 - \pi/4\right]$

$= \frac{\pi}{4}$

d iii) $Var(X) = \int x^2 f(x)\,dx - \mu^2$

$\int x^2 f(x)\,dx = \frac{1}{2}\int x^2\sin(x+\pi/4)\,dx$

$= \frac{1}{2}\left[-x^2\cos\left(x+\frac{\pi}{4}\right) + \int 2x\cos\left(x+\frac{\pi}{4}\right)dx\right]$

$= \frac{1}{2}\left[-x^2\cos\left(x+\frac{\pi}{4}\right) + 2x\sin\left(x+\frac{\pi}{4}\right) - \int 2\sin\left(x+\frac{\pi}{4}\right)dx\right]$

$= \frac{1}{2}\left[-x^2\cos\left(x+\frac{\pi}{4}\right) + 2x\sin\left(x+\frac{\pi}{4}\right) + 2\cos\left(x+\frac{\pi}{4}\right)\right]_{-\pi/4}^{3\pi/4}$

$= \frac{1}{2}\left[\frac{-9\pi^2}{16}(-1) + 0 + 2(-1) - \left[-\frac{\pi^2}{16}(1) + 0 + 2\right]\right]$

$= \frac{1}{2}\left[\frac{9\pi^2}{16} - 2 + \frac{\pi^2}{16} - 2\right]$

$= \frac{1}{2}\left[\frac{5\pi^2}{8} - 4\right]$

$= \frac{5\pi^2}{16} - 2$

$\therefore Var(X) = \frac{5\pi^2}{16} - 2 - \left(\frac{\pi}{4}\right)^2$

$= \frac{\pi^2}{4} - 2$

Question 14

a) P: I am beautiful and tall

$\neg P$: I am not tall or I am not beautiful

$\neg P$ I am not beautiful or I am not tall — either

b) $z = \text{cis}\theta$

$z^n = \text{cis}\, n\theta \qquad 1/z^n = z^{-n} = \text{cis}(-n\theta)$

$z^n + 1/z^n = \text{cis}\, n\theta + \text{cis}(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$\cos(-\alpha) = \cos\alpha$ is even

$\sin(-\alpha) = -\sin\alpha$ is odd

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

ii) $(z + 1/z)^6$

$= z^6 + 6z^5 \cdot \frac{1}{z} + 15z^4 \cdot \frac{1}{z^2} + 20z^3 \cdot \frac{1}{z^3} + 15z^2 \frac{1}{z^4} + 6z \cdot \frac{1}{z^5} + \frac{1}{z^6}$

$= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$

$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$

$= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

iii) $(z + 1/z)^6 = (2\cos\theta)^6 = 64\cos^6\theta$

iii) $\therefore \int_0^{\pi/6} \cos^6\theta\, d\theta = \frac{1}{64}\int_0^{\frac{\pi}{6}} 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20\, d\theta$

$= \frac{1}{64}\left[\frac{1}{3}\sin 6\theta + 3\sin 4\theta + 15\sin 2\theta + 20\theta\right]_0^{\pi/6}$

$= \frac{1}{64}\left[\frac{1}{3}\sin\pi + 3\sin\frac{2\pi}{3} + 15\sin\frac{\pi}{3} + \frac{20\pi}{6} - (0)\right]$

$= \left(0 + 3\times\sqrt{3}/2 + 15 \cdot \sqrt{3}/2 + 10\pi/3\right) \times 1/64$

$= \left(9\sqrt{3} + \frac{10\pi}{3}\right) \cdot \frac{1}{64} = \frac{9\sqrt{3}}{64} + \frac{5\pi}{96}$

14 c) $\ddot{x} = -4(x+2)$ centre $\ddot{x} = 0$

$x = -2$

$n = 2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x - 8$

$\frac{1}{2}v^2 = \int -4x - 8\, dx$

$\frac{1}{2}v^2 = -2x^2 - 8x + c \qquad v = 0 \quad x = 2$

$\frac{1}{2}(4) = -2(4) - 8(2) + c$

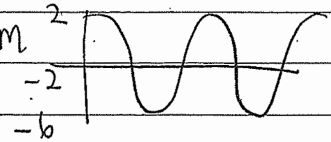
$c = 24$

$\therefore \frac{1}{2}v^2 = -2x^2 - 8x + 24$

$v^2 = -4x^2 - 16x + 48$

$v^2 = 48 - 16x - 4x^2$ QED

(ii) SHM



$a = 4$

$n = 2$

$\begin{pmatrix} v = 0 \\ x = 2 \\ t = 0 \end{pmatrix}$

modelled by equation $x = a\cos(nt + \alpha) - 2$

$x = 4\cos(2t + \alpha) - 2 \qquad x = 2,\ t = 0,\ \alpha = 0$

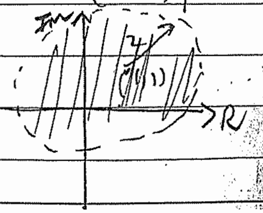
$\therefore x = 4\cos(2t) - 2$

d) $\sqrt{2}e^{\pi/4 i} = \sqrt{2}\,\text{cis}\frac{\pi}{4} = \sqrt{2}\left(\cos\pi/4 + i\sin\pi/4\right)$

$= \sqrt{2}\left(1/\sqrt{2} + i\ 1/\sqrt{2}\right)$

$= 1 + i$

$\therefore |z - (1+i)| < 2$



Question 15.

a) i) $\int_0^a f(x)\,dx$ let $x = a - u$ $\quad x = a \quad u = 0$

$dx = -du \quad x = 0 \quad u = a$

$\therefore \int_a^0 f(a-u) \cdot -du$

$-\int_a^0 f(a-u)\,du$

$\int_0^a f(a-u)\,du = \int_0^a f(a-x)\,dx$

Should say. as definite $\int$ is independent of variable used "dummy" variable

ii) $\int_0^1 x^2\sqrt{1-x}\,dx$

$\int_0^1 (1-x)^2\sqrt{1-(1-x)}\,dx$

$= \int_0^1 (1-2x+x^2)\sqrt{x}\,dx$

$= \int_0^1 \sqrt{x} - 2x\sqrt{x} + x^2\sqrt{x}\,dx$

$= \int_0^1 x^{1/2} - 2x^{3/2} + x^{5/2}\,dx$

$= \frac{2}{3}x^{3/2} - \frac{4}{5}x^{5/2} + \frac{2}{7}x^{7/2}\Big]_0^1$

$= 2/3 - 4/5 + 2/7 - (0)$

$= 16/105$

b) $F = m\ddot{x} = 16 - 4v^2 \quad m = 4kg \quad \ddot{x} = v\frac{dv}{dx}$

$4v\frac{dv}{dx} = 16 - 4v^2$

$\frac{dv}{dx} = \frac{4-v^2}{v} \quad \therefore \frac{dx}{dv} = \frac{v}{4-v^2}$

$x = -\frac{1}{2}\ln|4-v^2| + C \quad x=0,\ v=0 \quad C = \frac{1}{2}\ln 4$

$\therefore x = \frac{1}{2}\ln 4 - \frac{1}{2}\ln|4-v^2|$

ii) $F = 4\ddot{x} = 16 - 4v^2$

$\ddot{x} = 4 - v^2$

$\frac{dv}{dt} = 4 - v^2$

$\int_0^t dt = \int_0^v \frac{1}{4-v^2}\,dv$

$t = \int_0^v \frac{1}{(2-v)(2+v)}\,dv$

$\frac{1}{(2-v)(2+v)} = \frac{A}{2-v} + \frac{B}{2+v}$

$1 = A(2+v) + B(2-v)$

let $v = -2$

$1 = B(4) \quad B = 1/4$

let $v = 2$

$1 = A(4) \quad A = 1/4$

$= \int_0^v \frac{1/4}{2-v} + \frac{1/4}{2+v}\,dv$

$= \frac{1}{4}\int_0^v \frac{1}{2-v} + \frac{1}{2+v}\,dv$

$= \frac{1}{4}\Big[-\ln|2-v| + \ln|2+v|\Big]_0^v$

$= \frac{1}{4}\Big[-\ln|2-v| + \ln|2+v| - (-\ln 2 + \ln 2)\Big]$

$4t = \ln\left|\frac{2+v}{2-v}\right|$

$e^{4t} = \frac{2+v}{2-v}$

$(2-v)e^{4t} = 2 + v$

$2e^{4t} - ve^{4t} = 2 + v$

$2e^{4t} - 2 = v(1 + e^{4t})$

$v = \frac{2(e^{4t}-1)}{1+e^{4t}}$

$t = 3 \quad v = \frac{2(e^{12}-1)}{1+e^{12}} \rightarrow$ Now find x using (1)

$x = \frac{1}{2}\ln 4 - \frac{1}{2}\ln|4-v^2|$

$v^2 \approx 3.9999$ $\quad \doteq 5.31$

15c)

test $n=1$

$= 2\left(\dfrac{1+(-1/3)^1}{1-(-1/3)}\right)$

$= 2\left(\dfrac{2/3}{4/3}\right)$

$= 2 \times 1/2$

$= 1$

$= x_1$ as given ∴ the statement is true for $n=1$

Assume true for $n=k$

ie $x_k = 2 \times \left(\dfrac{1+\alpha^k}{1-\alpha^k}\right)$ $\alpha = -1/3$

Test for $n=k+1$

ie $x_{k+1} = \dfrac{4+x_k}{1+x_k}$ Now use the assumption.

$= \dfrac{4 + 2\left(\dfrac{1+\alpha^k}{1-\alpha^k}\right)}{1 + 2\left(\dfrac{1+\alpha^k}{1-\alpha^k}\right)}$

$= \dfrac{4(1-\alpha^k) + 2(1+\alpha^k)}{1-\alpha^k + 2(1+\alpha^k)}$

$= \dfrac{4 - 4\alpha^k + 2 + 2.\alpha^k}{1-\alpha^k + 2 + 2\alpha^k}$

$= \dfrac{6 - 2\alpha^k}{3+\alpha^k}$

$= 2\left(\dfrac{3-\alpha^k}{3+\alpha^k}\right)$

$2\left[\dfrac{3-\alpha^k}{3+\alpha^k}\right]$ $\begin{matrix}\div -3 \\ \div -3\end{matrix}$ $\alpha = \frac{-1}{3}$

$2\left[\dfrac{-1-(-1/3)^k(-1/3)}{-1+(-1/3)^k(-1/3)}\right]$

$= 2\left[\dfrac{-1-(-1/3)^{k+1}}{-1+(-1/3)^{k+1}}\right]$

$= 2\left[\dfrac{-(1+(-1/3)^{k+1})}{-1(1-(-1/3)^{k+1})}\right]$

$= 2\left[\dfrac{1+\alpha^{k+1}}{1-\alpha^{k+1}}\right]$

which is in the form required ∴ If $n=k$ it is also true for $n=k+1$. As true for $n=1$, it is also true for $n=2,3,4\ldots$

∴ Hence true for all $n \in \mathbb{Z}^+$.

Now

(ii) $x_n \to 2$ as $n \to \infty$

as $\alpha^n \to 0$ as $\alpha = \frac{-1}{3}$

Question 16.

a) consider cases not ÷ by 3 is remainder 1 or 2

∴ consider cases of $n = 3p+1$ or $3p-1$. $p \in \mathbb{Z}^+$

could test $n = 3p+1$ & $n = 3p+2$ $p \geq 0$

Case 1 $n = 3p+1$ $p \in \mathbb{Z}^+$

$n^2 + 6n + 2$

$= (3p+1)^2 + 6(3p+1) + 2$

$= 9p^2 + 6p + 1 + 18p + 6 + 2$

$= 9p^2 + 24p + 9$

$= 3(p^2 + 8p + 3)$

$= 3q$ where $q = p^2 + 8p + 3 \in \mathbb{Z}$

which is ÷ by 3

Case 2

$n = 3p - 1$ $p \in \mathbb{Z}^+$

$n^2 + 6n + 2$

$= (3p-1)^2 + 6(3p-1) + 2$

$= 9p^2 - 6p + 1 + 18p - 6 + 2$

$= 9p^2 + 12p + 3 - 6$

$= 9p^2 + 12p - 3$

$= 3(3p^2 + 4p - 1)$

$= 3m$ $m = 3p^2 + 4p - 1 \in \mathbb{Z}^+$

which is ÷ by 3

∴ if n is not divisible by 3 then $n^2 + 6n + 2$ is divisible by 3.

b) $I_n = \int_0^1 (1-x^2)^n \, dx$ $J_n = \int_0^1 x^2(1-x^2)^n \, dx$

i) $I_n = \int_0^1 1(1-x^2)^n \, dx$

$= \left[(1-x^2)^n . x\right]_0^1 - \int_0^1 n(1-x^2)^{n-1} . -2x . x \, dx$

$= \left[(1-x^2)^n . x\right]_0^1 + 2n \int_0^1 x^2(1-x^2)^{n-1} \, dx$

$= 0 - 0 + 2n J_{n-1}$

$= 2n J_{n-1}$

ii) $\frac{1}{2n} I_n = J_{n-1}$

$\frac{1}{2n} I_n = \int_0^1 x^2(1-x^2)^{n-1} \, dx$ change x^2 into $-(1-x^2-1)$

$= \int_0^1 -(1-x^2-1)(1-x^2)^{n-1} dx$

$= \int_0^1 \left[1-(1-x^2)\right](1-x^2)^{n-1} dx$

$= \int_0^1 (1-x^2)^{n-1} - \int_0^1 (1-x^2)^n \, dx$

$= I_{n-1} - I_n$

$\left(\frac{1}{2n} + 1\right) I_n = I_{n-1}$

$\frac{1+2n}{2n} I_n = I_{n-1}$

$I_n = \frac{2n}{1+2n} I_{n-1}$

Show that $J_n = I_n - I_{n+1}$

$$J_n = \int_0^1 x^2(1-x^2)^n \, dx$$

$$= \int_0^1 -(1-x^2-1)(1-x^2)^n \, dx$$

$$= \int_0^1 \left[-(1-x^2)+1\right](1-x^2)^n \, dx$$

$$= \int_0^1 -(1-x^2)^{n+1} + \int_0^1 (1-x^2)^n \, dx$$

$$J_n = -I_{n+1} + I_n$$

$$\therefore J_n = I_n - I_{n+1} \longleftarrow ※$$

OR

$$RHS = I_n - I_{n+1}$$

$$= \int_0^1 (1-x^2)^n dx - \int_0^1 (1-x^2)^{n+1}$$

$$= \int_0^1 (1-x^2)^n\left[1-(1-x^2)\right] dx$$

$$= \int_0^1 (1-x^2)^n (x^2) \, dx$$

$$= \int_0^1 x^2(1-x^2)^n \, dx$$

$$= J_n$$

$$= LHS.$$

iv) Now $I_{n+1} = \dfrac{2(n+1)}{2(n+1)+1} I_n$ using part (ii)

$$= \frac{2n+2}{2n+3} I_n$$

※ use this one

$$I_{n+1} = \frac{2n}{2n+3} I_n + \frac{2}{2n+3} I_n$$

sub into part (iii) ※

$$J_n = I_n - \frac{2n+2}{2n+3} I_n$$

$$= \left(1 - \frac{2n+2}{2n+3}\right) I_n$$

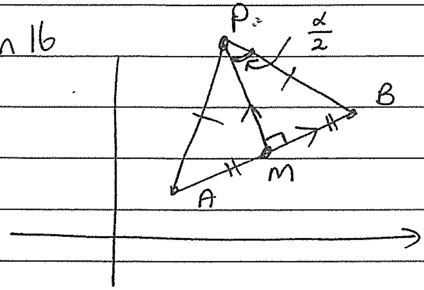
$$= \frac{2n+3-2n-2}{2n+3} I_n$$

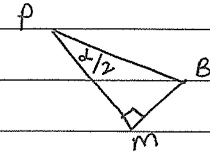
$$= \frac{1}{2n+3} I_n \quad \text{QED.}$$

IV) $\dfrac{1}{2n+3} 2n J_{n-1}$

$$= \frac{2n}{2n+3} J_{n-1}$$

Question 16





$$\tan\frac{\alpha}{2} = \frac{|\vec{MB}|}{|\vec{MP}|}$$

$$\therefore \tan\frac{\alpha}{2}|\vec{MP}| = |\vec{MB}|$$

$$\therefore |\vec{MP}| = \cot\frac{\alpha}{2}|\vec{MB}|$$

$\therefore \cot\frac{\alpha}{2}$ is scalar X

$$\vec{AB} = z_2 - z_1$$

$$\therefore \vec{AM} = \vec{MB} = \frac{1}{2}(z_2 - z_1)$$

$\vec{MP}$ = rotation by 90° of $\vec{MB}$ but also scalar X of $\vec{MB}$

$$\vec{MP} = i\cot\frac{\alpha}{2} \cdot \frac{1}{2}(z_2 - z_1)$$

ii) $\vec{OP} = \vec{OA} + \vec{AM} + \vec{MP}$

$$= z_1 + \frac{1}{2}(z_2 - z_1) + \frac{1}{2} i\cot\frac{\alpha}{2}(z_2 - z_1)$$

$$= z_1 + \frac{1}{2}z_2 - \frac{1}{2}z_1 + \frac{1}{2}i\cot\frac{\alpha}{2} z_2 - \frac{1}{2}i\cot\frac{\alpha}{2} z_1$$

$$= \left(1 - \frac{1}{2} - \frac{1}{2}i\cot\frac{\alpha}{2}\right)z_1 + \left(\frac{1}{2} + \frac{1}{2}i\cot\frac{\alpha}{2}\right)z_2$$

$$= \frac{1}{2}\left(1 - i\cot\frac{\alpha}{2}\right)z_1 + \frac{1}{2}\left(1 + i\cot\frac{\alpha}{2}\right)z_2.$$

≈